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Abstract

The development of adaptive observer techniques for nonlinear systems in the output canonical form is proposed applying additional impulsive feedback in the observer equations. The stability of new impulsive adaptive observer is investigated. It is shown that under some conditions the proposed impulsive feedback can improve the rate of the observer convergence or relax the requirement on persistency of excitation, which is usually introduced to ensure convergence of the parameter estimates. The proposed feasibility conditions include positivity of dwell time (boundedness of impulse frequency) and solvability of some Lyapunov-like matrix inequalities. The results are illustrated by simulation for three examples (including a single link flexible joint robot example).

1 INTRODUCTION

Design of adaptive observers for nonlinear systems was extensively studied during the last two decades after [2]. Such an interest was particularly motivated by possible application of observers to information encoding and transmission. Typically a chaotic dynamical system is used as a transmitter and its output signal is changed by its parameter modulation (see special issues [16] and [17]).

It was shown in [9] that it is possible to build a receiver based on adaptive observers, which can track output of transmitter and estimate transmitter parameters under passivity conditions. Several techniques were proposed to design the adaptive observers [7, 8, 3, 22, 24], for the most part of them based on passifiability property of the observed system under the relative degree one assumption. Other related solutions can be found in [14, 18], where a state feedback was used for adaptive control construction. The papers [10, 32, 33] overcame the relative degree limitation for adaptive observer-based communication systems and extended them to a class of nonpassifiable systems. Other application areas are nonlinear systems synchronization and control [3, 4, 13, 28, 31]. The common applicability conditions of the adaptive observer approach include the output canonical form for the observed system (recently the approach has been extended to a more general case in [6]) and persistency of excitation conditions. The former property is hard to guarantee and it is used to prove the convergence of parameter estimates to their ideal values [10, 32, 33].

The purpose of this work is to relax the applicability conditions via additional impulsive feedbacks. Recently this idea has been used to improve convergence rate and quality of estimation of the state observers for nonlinear Lipschitz systems [29], where the conventional correction term $L[y(t) - \hat{y}(t)]$ [20, 27] has been replaced by the augmented one $L[y(t)-\hat{y}(t)]+\sum_{k=1}^{+\infty} K[y(t)-\hat{y}(t)]\delta(t-t_k)$ ($\delta(t)$ is the Dirac impulse at the time instant $t = 0$). In the work [1] an adaptive impulsive observer is studied where the impulses are originated by the sampled measurements. In the present paper the idea of [29] is borrowed to deal with the adaptive observer design for the systems presentable in the output canonical form. As a side of result, the proposed impulsive excitation allows the achievable in the system level of persistency of excitation to be improved.

The more detailed problem statement is given in Section 2. The main results are presented in Section 3. The persistency of excitation issue is discussed in Section 4. The examples of computer simulation illustrate advantages of the proposed impulsive adaptive observers in Section 5.

2 PROBLEM STATEMENT

Consider the nonlinear system

$$\dot{x} = A(y)x + \phi(y,u) + G(y)\theta + d, \; y = Cx, \; y_v = y + v, \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$, $\theta \in \mathbb{R}^q$ are respectively the state, the output, the control input and the vector of unknown parameters; $d \in \mathbb{R}^n$, $v \in \mathbb{R}^p$ are the disturbances and the measurement noise; the functions $A : \mathbb{R}^p \to \mathbb{R}^{n\times n}$, $\phi : \mathbb{R}^{p+m} \to \mathbb{R}^n$ and $G : \mathbb{R}^p \to \mathbb{R}^{n\times q}$ are locally Lipschitz continuous; $y_v \in \mathbb{R}^p$ represents the available for a designer vector of noisy measurements. The Euclidean norm is denoted as $|x|$, for a matrix $|\cdot|$ gives the corresponding induced norm, and the symbol $||u||_{[t_0,t)}$ is stated for the L_∞ norm of the input $u(t)$ ($u : \mathbb{R}_+ \to \mathbb{R}^m$ is assumed to be a measurable and locally essentially bounded function of time $t \geq 0$, as well as other inputs d, v):

$$||u||_{[t_0,T)} = \text{ess} \sup_{t_0 \leq t < T} |u(t)|,$$

if $T = +\infty$ then we will simply write $||u||$. Under introduced conditions the system (1) for any initial condition $x_0 \in \mathbb{R}^n$ has the unique absolutely continuous solution $x(t, x_0)$ defined at least locally in time $t \geq 0$ (the symbol $x(t)$ is used to denote the system (1) solutions if the origin of initial conditions is clear form the context). In this work we will assume that all signals in (1) are bounded.

Assumption 1. *$||u|| < +\infty$, $||v|| < +\infty$, $||d|| < +\infty$ and there is a set $X \subset \mathbb{R}^n$ such that $||x(t, x_0)|| < +\infty$ for all $x_0 \in X$.*

Note that the set X could be unknown for a designer. Owing to these conditions there exist constants $L_A > 0$, $L_\phi > 0$ and $L_G > 0$ (generically dependent on the initial conditions x_0 in the system (1) and the matrix C) such that

$$|A(y) - A(y_v)| \leq L_A|v|, |\phi(y, u) - \phi(y_v, u)| \leq L_\phi|v|, |G(y) - G(y_v)| \leq L_G|v|.$$

In addition, in this case there is a compact set $Y \subset \mathbb{R}^p$ such that $y_v \in Y$ for all $t \geq 0$. For a matrix $P \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote the maximum and the minimum eigenvalues respectively.

Taking in mind Assumption 1 and other auxiliary conditions introduced above it is required to design an adaptive observer that has to provide the state x and the parameters θ estimation in the case $d = v \equiv 0$. For the common case $||v|| < +\infty$, $||d|| < +\infty$ the estimates have to be bounded.

3 MAIN RESULTS

This section has four parts. First, some preliminary results dealing with persistency of excitation are introduced. Second, the impulsive adaptive observer equations are presented. Third, the observer stability is proven. Finally, a restricted class of systems is analyzed whose stability conditions can be formulated in terms of LMIs.

3.1 *Persistency of excitation condition*

The Lebesgue measurable and square integrable matrix function $R : \mathbb{R} \to \mathbb{R}^{l\times k}$ with the dimension $l \times k$ admits (ℓ, ϑ)–Persistency of Excitation (PE) condition, if there exist constants $\ell > 0$ and $\vartheta > 0$ such that

$$\int_t^{t+\ell} R(s)R(s)^T ds \geq \vartheta I_l$$

for any $t \in \mathbb{R}_+$, where I_l denotes the identity matrix of dimension $l \times l$. Such a matrix function R is called PE if there are some $\ell > 0$, $\vartheta > 0$ such that it is (ℓ, ϑ)-PE.

Lemma 1. *Consider a time-varying linear dynamical system*

$$\dot{p} = -\gamma R(t)R(t)^T p + b(t),\ t_0 \in \mathbb{R}_+, \tag{2}$$

where $p \in \mathbb{R}^l$, $\gamma > 0$ and the function $R : \mathbb{R}_+ \to \mathbb{R}^{l\times k}$ is continuous and bounded (i.e. $\rho = \sup_{t\geq 0}|R(t)|^2 < +\infty$), $b : \mathbb{R}_+ \to \mathbb{R}^l$ is Lebesgue measurable and essentially bounded, function R is (ℓ,ϑ)–PE for some $\ell > 0$, $\vartheta > 0$. Then, for any initial condition $p(t_0) \in \mathbb{R}^l$, the solution $p(t)$ of the system is defined for all $t \geq t_0$ and satisfies the following inequality

$$|p(t)| \leq \sqrt{\rho\alpha}[e^{-0.5\gamma\alpha^{-1}(t-t_0)}|p(t_0)| + \gamma^{-1}\alpha||b||]$$

for $\alpha = \gamma\eta^{-1}e^{2\eta\ell}$ and $\eta = -0.5\ell^{-1}\ln(1 - \frac{\gamma\vartheta}{1+\gamma^2\ell^2\rho^2})$.

Proof. Since $|\dot{p}| \leq \gamma\rho|p| + ||b||$, the solutions $p(t)$ are defined for all $t \geq t_0$ for all initial conditions $p(t_0) \in \mathbb{R}^l$. For the case of absence of the external input b the system (2) can be reduced to the following linear time-varying autonomous system:

$$\dot{p}(t) = A(t)p(t),\ A(t) = A(t)^T = -\gamma R(t)R(t)^T, \tag{3}$$

where $-\gamma\rho I_l \leq A(t) \leq 0$ is a continuous and bounded matrix function, $0 < \rho = \sup_{t\geq 0}|R(t)|^2 < +\infty$ and $|\cdot|$ is L_2 induced matrix norm. Since the matrix A is negative semidefinite and symmetric, the property $|p(\tau)| \geq |p(t)|$ holds for all $t \geq \tau \geq 0$, i.e. the system (3) is Lyapunov stable. For the initial conditions $p(t_0)$ the system (3) has the solution $p(t) = \Phi(t,t_0)p(t_0)$ for $t \geq t_0$, where $\Phi(t,t_0)$ is the state-transition matrix of (3) [19]. According to the exposition above $e^{-\gamma\rho(t-t_0)}I_l \leq \Phi(t,t_0) \leq I_l$ for $t \geq t_0$, and our goal is to show that under PE condition the upper estimate for $\Phi(t,t_0)$ can be replaced with an exponentially decreasing one.

Consider for the system (3) a Lyapunov function $W(p) = 0.5p^Tp$:

$$\dot{W}(t) = p(t)^TA(t)p(t) \leq 0,\ t \geq 0,$$

and let us show that the system (3) is globally exponentially stable. For this purpose note that $0.5|a|^2 \leq |b|^2 + |a-b|^2$ for any $a, b \in \mathbb{R}^l$ and

$$\sup_{t\in[t_0,t_0+\ell]}|p(t_0) - p(t)| = \sup_{t\in[t_0,t_0+\ell]}\left|\int_{t_0}^{t}\dot{p}(\tau)d\tau\right|$$
$$\leq \sup_{t\in[t_0,t_0+\ell]}\int_{t_0}^{t}|\dot{p}(\tau)|d\tau = \int_{t_0}^{t_0+\ell}|\dot{p}(\tau)|d\tau,$$

the Jensen's inequality for convex functions gives

$$\left[\int_{t_0}^{t_0+\ell}|\dot{p}(t)|dt\right]^2 \leq \ell\int_{t_0}^{t_0+\ell}|\dot{p}(t)|^2dt,$$

and

$$\int_{t_0}^{t_0+\ell} |\dot{p}(t)|^2 dt = \gamma^2 \int_{t_0}^{t_0+\ell} p(t)^T R(t)R(t)^T R(t)R(t)^T p(t)dt \leq \gamma^2 \rho \int_{t_0}^{t_0+\ell} p(t)^T R(t)R(t)^T p(t)dt.$$

Then we obtain

$$\begin{aligned}\int_{t_0}^{t_0+\ell} p(t)^T R(t)R(t)^T p(t)dt &\geq 0.5 \int_{t_0}^{t_0+\ell} p(t_0)^T R(t)R(t)^T p(t_0)dt \\ &- \int_{t_0}^{t_0+\ell} [p(t_0)-p(t)]^T R(t)R(t)^T [p(t_0)-p(t)]dt \\ &\geq 0.5\vartheta p(t_0)^T p(t_0) - \rho \int_{t_0}^{t_0+\ell} [p(t_0)-p(t)]^T [p(t_0)-p(t)]dt \\ &\geq 0.5\vartheta p(t_0)^T p(t_0) - \ell\rho \sup_{t\in[t_0,t_0+\ell]} |p(t_0)-p(t)|^2 \\ &\geq 0.5\vartheta p(t_0)^T p(t_0) - \ell\rho \left[\int_{t_0}^{t_0+\ell} |\dot{p}(t)|dt\right]^2 \\ &\geq 0.5\vartheta p(t_0)^T p(t_0) - \ell^2\rho \int_{t_0}^{t_0+\ell} |\dot{p}(t)|^2 dt \\ &\geq 0.5\vartheta p(t_0)^T p(t_0) - \gamma^2\ell^2\rho^2 \int_{t_0}^{t_0+\ell} p(t)^T R(t)R(t)^T p(t)dt,\end{aligned}$$

and finally

$$\int_{t_0}^{t_0+\ell} p(t)^T R(t)R(t)^T p(t)dt \geq \frac{\vartheta}{1+\gamma^2\ell^2\rho^2} W(t_0).$$

Therefore, $W(t_0+\ell)-W(t_0) \leq -\frac{\gamma\vartheta}{1+\gamma^2\ell^2\rho^2} W(t_0)$ and $W(t_0+\ell) \leq (1-\frac{\gamma\vartheta}{1+\gamma^2\ell^2\rho^2})W(t_0)$ for any $t_0 \geq 0$, then $W(t_0+\ell) \leq e^{-2\eta\ell} W(t_0)$ for $\eta = -0.5\ell^{-1}\ln(1-\frac{\gamma\vartheta}{1+\gamma^2\ell^2\rho^2})$ and for solutions of the system (3) the inequalities are satisfied

$$e^{-\gamma\rho(t-t_0)}|p(t_0)| \leq |p(t)| \leq e^{-\eta(t-t_0-\ell)}|p(t_0)| \quad \forall t \geq t_0.$$

The system (3) is uniformly exponentially stable and $e^{-\gamma\rho(t-t_0)} \leq |\Phi(t,t_0)| \leq e^{-\eta(t-t_0-\ell)}$ for $\forall t \geq t_0$.

Next, following the converse Lyapunov theorem for such a kind of systems [19], the matrix $P(t) = \int_t^{+\infty} \Phi(\tau,t)^T \Phi(\tau,t) d\tau$ satisfies the differential equation $\dot{P}(t)+P(t)A(t)+A(t)^T P(t)+I_l = 0$ for all $t \geq 0$, and $V(t,p) = p^T P(t) p$ is a Lyapunov function for the system (3). Since $0.5\gamma^{-1}\rho^{-1} I_l \leq \int_t^{+\infty} e^{-2\gamma\rho(\tau-t)} d\tau I_l \leq P(t) \leq \int_t^{+\infty} e^{-2\eta(\tau-t-\ell)} d\tau I_l \leq 0.5\eta^{-1} e^{2\eta\ell} I_l$ we have for all $p \in \mathbb{R}^l$:

$$0.5\gamma^{-1}\rho^{-1}|p|^2 \leq V(t,p) \leq 0.5\eta^{-1} e^{2\eta\ell}|p|^2.$$

Consider this Lyapunov function V for the system (2) (we will use the fact that $2p^T P(t)b(t) \leq \varrho p^T P(t)p + \varrho^{-1} b(t)^T P(t)b(t)$ for any $\varrho > 0$):

$$\begin{aligned} \dot{V} &= -p^T p + 2p^T P(t)b(t) \\ &\leq -\eta e^{-2\eta\ell} V + \eta^{-1} e^{2\eta\ell} b(t)^T P(t)b(t) \\ &\leq -\eta e^{-2\eta\ell} V + 0.5\eta^{-2} e^{4\eta\ell} ||b||^2. \end{aligned}$$

Then for all $t \geq t_0$ the following inequality holds:

$$|p(t)| \leq \sqrt{\gamma\rho\eta^{-1}} e^{\eta\ell} [e^{-0.5\eta e^{-2\eta\ell}(t-t_0)} |p(t_0)| + \eta^{-1} e^{2\eta\ell} ||b||].$$

$\square$

This lemma states that a linear system with a persistently excited time-varying matrix gain and a bounded additive disturbance has bounded solutions with exponentially converging bound for transient mode, see Lemma 1 in [23] or the monograph [26] for related results.

3.2 *Impulsive adaptive observer*

Inspired by [29] we model the influence of impulses on the adaptive observer dynamics as an initial condition shift at the instant of the impulse. Thus the observer equations have two parts. The first one represents the continuous dynamics proposed in [10, 32, 33], the second part is event-based and it describes the discontinuous jump of initial conditions at the instants of impulses t_k, $k = 1, 2, \ldots$:

$$\begin{cases} \dot{z}(t) = & A[y_v(t)]z(t) + \phi[y_v(t), u(t)] + G[y_v(t)]\hat{\theta}(t) \\ & +L[y_v(t)]\{y_v(t) - Cz(t)\} - \Omega(t)\dot{\hat{\theta}}(t), \\ \dot{\Omega}(t) = & \{A[y_v(t)] - L[y_v(t)]C\}\Omega(t) - G[y_v(t)], \\ \dot{\hat{\theta}}(t) = & -\gamma\Omega(t)^T C^T \{y_v(t) - Cz(t)\}, \end{cases} \quad t \in [t_{k-1}, t_k); \tag{4}$$

$$\begin{cases} z(t_k^+) = & F[y_v(t_k)]z(t_k) + K[y_v(t_k)]\{y_v(t_k) - Cz(t_k)\}, \\ \Omega(t_k^+) = & \{F[y_v(t_k)] - K[y_v(t_k)]C\}\Omega(t_k), \end{cases} \quad t = t_k, \tag{5}$$

where $z \in \mathbb{R}^n$ is the estimate of x, $\hat{\theta} \in \mathbb{R}^q$ is the estimate of θ, $\Omega \in \mathbb{R}^{n\times q}$ is the auxiliary filtering variable; $L : \mathbb{R}^p \to \mathbb{R}^{n\times p}$, $K : \mathbb{R}^p \to \mathbb{R}^{n\times p}$ and $F : \mathbb{R}^p \to \mathbb{R}^{n\times n}$ are the locally Lipschitz continuous functional gains of the observer (their values will be specified in the next section); the initial conditions $z(t_0^+) = z_0 \in \mathbb{R}^n$, $\Omega(t_0^+) = \Omega_0 \in \mathbb{R}^{n\times q}$, $\hat{\theta}(t_0) = \hat{\theta}_0 \in \mathbb{R}^q$, $\gamma > 0$ is a design parameter. It is assumed that t_k, $k = 0, 1, \ldots$ is a strictly increasing time sequence satisfying $0 = t_0 < t_1 < t_2 < \ldots$ and

$$z(t_k^+) = \lim_{h\to 0} z(t_k - h), \quad \Omega(t_k^+) = \lim_{h\to 0} \Omega(t_k - h).$$

Note that for a continuous signal $y(t_k) = y(t_k^+) = \lim_{h\to 0} y(t_k - h)$. It is assumed that the number of impulses (and, hence, the number of jumps in (5)) can be finite (then $k = \overline{0, N}$, $0 \leq N < +\infty$) or infinite ($\lim_{k\to+\infty} t_k = +\infty$).

According to the equations (4), (5) the variable $\hat{\theta}$ has continuous dynamics, the discontinuity appears in the variables z and Ω only. As it has been observed in [29] the impulsive event-based part (5) can be easily realized implementing the observer (4), (5) on a microcontroller, since there any variable can be set to a desired value at any sampling instant of time.

3.3 *Proof of estimation abilities*

As in the switched systems theory [21] we will say that the sequence of instants t_k, $k = 0, 1, \ldots$ has the dwell-time $\tau_D > 0$ if the inequality $t_k - t_{k-1} \geq \tau_D$ is satisfied for all $k = 1, 2, \ldots$. Note that positivity of dwell time condition is equivalent to boundedness of frequency of impulses [15].

The first theorem presents the most general conditions of the impulsive adaptive observer stability.

Theorem 1. *Let Assumption 1 be satisfied, the signals $G[y_v(t)]^T$ and $\Omega(t)^T C^T$ be (ℓ, ϑ)–PE for some $\ell > 0$, $\vartheta > 0$ and the sequence of instants t_k, $k = 0, 1, \ldots$ have a dwell-time $\tau_D > 0$. If there exist a matrix $P = P^T > 0$ with the dimension $n \times n$ and the locally Lipschitz continuous matrix functions $L : \mathbb{R}^p \to \mathbb{R}^{n\times p}$, $K : \mathbb{R}^p \to \mathbb{R}^{n\times p}$, $F : \mathbb{R}^p \to \mathbb{R}^{n\times n}$ such that*

$$[A(y_v) - L(y_v)C]^T P + P[A(y_v) - L(y_v)C] \leq -\nu P,\ \nu > 0,$$
$$[F(y_v) - K(y_v)C]^T P[F(y_v) - K(y_v)C] \leq 0.5\mu P,\ \mu > 0,$$

for all $y_v \in Y$ with $0 < \rho < 1$ where $\rho = \mu e^{-0.5\nu\tau_D}$, then:

1. *for any $x_0 \in X$, $z_0 \in \mathbb{R}^n$, $\Omega_0 \in \mathbb{R}^{n\times q}$, $\hat{\theta}_0 \in \mathbb{R}^q$ the solutions of the system (1)–(5) are bounded for all $t \geq 0$;*

2. *the following asymptotic estimates are satisfied:*

$$\lim_{t\to+\infty} |\theta - \hat{\theta}(t)| \leq \alpha^{1.5}|C|^2 S(||x||, ||v||, ||d||, |\theta|),$$
$$\lim_{t\to+\infty} |x(t) - z(t)| \leq H(||x||, ||v||, ||d||, |\theta|),$$

where

$$S(x,v,d,q) = D(F_{\max}x + K_{\max}v, \{L_A x + L_\phi + L_G q + L_{\max}\}v + d, v),$$
$$H(x,v,d,q) = Q(F_{\max}x + K_{\max}v, \{L_A x + L_\phi + L_G q + L_{\max}\}v + d, v),$$
$$L_{\max} = \sup_{y_v\in Y}|L(y_v)|,\ F_{\max} = \sup_{y_v\in Y}|I_n - F(y_v)|,\ K_{\max} = \sup_{y_v\in Y}|K(y_v)|,$$
$$G_{\max} = \sup_{y_v\in Y}|G(y_v)|,\ \omega(P) = \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)},$$
$$\chi = 2\nu^{-1}\omega(P)\{1+\omega(P)\sqrt{\mu/(1-\rho)}\},\ \alpha = \gamma\eta^{-1}e^{2\eta\ell},$$
$$\eta = -\frac{1}{2\ell}\ln(1 - \frac{\gamma\vartheta}{1+\gamma^2\ell^2(|C|\omega^2(P)\{|\Omega(t_0)| + \frac{2}{\nu}\{1+\sqrt{\frac{\mu}{1-\rho}}\}G_{\max}\})^2}),$$
$$Q(r,p,v) = \omega(P)^2\sqrt{2/(1-\rho)}r + \chi Z(\alpha^{1.5}|C|^2 D(r,p,v), E(r,p), \chi G_{\max}, v, p),$$
$$Z(q,d,W,v,p) = G_{\max}q + \gamma W^2|C|\{|C|(d+Wq)+v\} + p,$$
$$D(r,p,v) = \chi^2 G_{\max}^2\{|C|E(r,p)+v\},\ E(r,p) = \omega(P)^2\sqrt{2/(1-\rho)}||r|| + \chi||p||.$$

Proof. Define the state estimation error $e = x - z$, the parametric estimation error $\tilde{\theta} = \theta - \hat{\theta}$ and the auxiliary error $\delta = e + \Omega\tilde{\theta}$, whose dynamics from (1) and (4) have the form for all $t \in [t_{k-1}, t_k)$, $k = 1, 2, ...$:

$$\begin{aligned}
\dot{e}(t) =& \{A[y_v(t)] - L[y_v(t)]C\}e(t) + G[y_v(t)]\tilde{\theta}(t) - \Omega(t)\dot{\tilde{\theta}}(t) + p(t), && (6)\\
p(t) =& \{A[y(t)] - A[y_v(t)]\}x(t) + \phi[y(t), u(t)] - \phi[y_v(t), u(t)] \\
&+ \{G[y(t)] - G[y_v(t)]\}\theta - L[y_v(t)]v(t) + d(t); \\
\dot{\delta}(t) =& \{A[y_v(t)] - L[y_v(t)]C\}\delta(t) + p(t); && (7)\\
\dot{\tilde{\theta}}(t) =& \gamma\Omega(t)^T C^T\{Ce(t) + v(t)\} && (8)\\
=& \gamma\Omega(t)^T C^T\{C\delta(t) - C\Omega(t)\tilde{\theta}(t) + v(t)\},
\end{aligned}$$

where $p \in \mathbb{R}^n$. Under Assumption 1 the property

$$||p|| \le \{L_A||x|| + L_\phi + L_G|\theta| + L_{\max}\}||v|| + ||d||,$$

holds (this input is bounded), and the equation (8) is actually valid for all $t \ge 0$ since the variable $\hat{\theta}$ has continuous dynamics only. The errors e and δ have impulse driven discrete dynamics, that from (5) can be written as follows for any t_k, $k = 0, 1, ...$ (note that $x(t_k) = x(t_k^+)$):

$$\begin{aligned}
e(t_k^+) = x(t_k^+) - z(t_k^+) =& x(t_k) - F[y_v(t_k)]z(t_k) && (9)\\
&- K[y_v(t_k)]\{y_v(t_k) - Cz(t_k)\} \\
=& \{F[y_v(t_k)] - K[y_v(t_k)]C\}e(t_k) + r(t_k), \\
r(t_k) =& (I_n - F[y_v(t_k)])x(t_k) - K[y_v(t_k)]v(t_k), \\
||r|| \le& F_{\max}||x|| + K_{\max}||v||;
\end{aligned}$$

$$\delta(t_k^+) = e(t_k^+) + \Omega(t_k^+)\tilde{\theta}(t_k^+) = \{F[y_v(t_k)] - K[y_v(t_k)]C\}\delta(t_k) + r(t_k), \quad (10)$$

where $r \in \mathbb{R}^n$ is another new disturbing bounded input. From the equations (4), (7), (10) it is possible to conclude that the variables Ω and δ have similar *hybrid* dynamics: in the continuous case it is a linear time-varying system with the matrix gain $A(y_v) - L(y_v)C$, the discrete linear time-varying dynamics is governed by $F(y_v) - K(y_v)C$. Both systems have bounded additive inputs. To prove boundedness of Ω and δ consider the Lyapunov function $V(\delta) = \delta^T P \delta$ for these variables (for brevity of presentation the variable δ is analyzed below). For all $t \in [t_{k-1}, t_k)$, $k = 1, 2, ...$ from (7) we have:

$$\begin{aligned}\dot{V}(t) =& \delta(t)^T(\{A[y_v(t)] - L[y_v(t)]C\}^T P + P\{A[y_v(t)] \\ & - L[y_v(t)]C\})\delta(t) + 2\delta(t)^T P p(t) \\ \leq & -\nu V(t) + 2\delta(t)^T P p(t) \leq -0.5\nu V(t) + 2\nu^{-1} p(t)^T P p(t),\end{aligned}$$

which gives the following estimate in the time domain:

$$V(t) \leq V(t_k) e^{-0.5\nu(t-t_k)} + 4\nu^{-2}\lambda_{\max}(P)||p||^2_{[t_k, t_{k+1})}. \tag{11}$$

According to (10) for all t_k, $k = 0, 1, ...$ the impulse based behavior of the function $V(\delta)$ yields:

$$\begin{aligned}V(t_k^+) = V[\delta(t_k^+)] =& 2\delta(t_k)^T\{F[y_v(t_k)] - K[y_v(t_k)]C\}^T P\{F[y_v(t_k)] \\ & - K[y_v(t_k)]C\}\delta(t_k) + 2r(t_k)^T P r(t_k) \\ \leq & \mu V(t_k) + 2r(t_k)^T P r(t_k) \leq \mu V(t_k) + 2\lambda_{\max}(P)|r(t_k)|^2.\end{aligned}$$

Therefore, on the closed interval $[t_k, t_{k+1}]$, $k = 0, 1, ...$ the Lyapunov function V behavior is described by:

$$\begin{aligned}V(t_{k+1}^+) \leq & \mu[V(t_k)e^{-0.5\nu(t_{k+1}-t_k)} + 4\nu^{-2}\lambda_{\max}(P)||p||^2_{[t_k, t_{k+1})}] \\ & + 2\lambda_{\max}(P)|r(t_k)|^2.\end{aligned} \tag{12}$$

Applying the estimate (12) recurrently $k+1$ times for $k = 0, 1, ...$ we obtain $(0 < \rho = \mu e^{-0.5\nu\tau_D} < 1)$:

$$\begin{aligned}V(t_{k+1}^+) \leq & \rho^{k+1} V(t_0) + 2\lambda_{\max}(P)[||r||^2 + 2\nu^{-2}\mu||p||^2]\sum_{i=0}^{k}\rho^i \\ \leq & \rho^{k+1} V(t_0) + 2\lambda_{\max}(P)(1-\rho)^{-1}[||r||^2 + 2\nu^{-2}\mu||p||^2].\end{aligned}$$

Passing back to the variables Ω and δ we get the estimates for all $k = 0, 1, ...$:

$$\begin{aligned}|\delta(t_{k+1}^+)| &\leq \omega(P)\{\rho^{0.5(k+1)}|\delta(t_0)| + \sqrt{2/(1-\rho)}[||r|| + \sqrt{2\mu}\nu^{-1}||p||]\}, \\ |\Omega(t_{k+1}^+)| &\leq \omega(P)\{\rho^{0.5(k+1)}|\Omega(t_0)| + 2\sqrt{\mu/(1-\rho)}\nu^{-1}G_{\max}\}.\end{aligned}$$

Since the inputs p and r are bounded, from (11) the variables Ω and δ have the same property for all $t \geq 0$, i.e.

$$\begin{aligned}|\delta(t)| \leq & \omega(P)\{\omega(P)\{\rho^{0.5k}|\delta(t_0)| + \sqrt{2/(1-\rho)}[||r|| \\ & + \sqrt{2\mu}\nu^{-1}||p||]\}e^{-0.25\nu(t-t_k)} + 2\nu^{-1}||p||\}, \\ |\Omega(t)| \leq & \omega(P)\{\omega(P)\{\rho^{0.5k}|\Omega(t_0)| + 2\sqrt{\mu/(1-\rho)}\nu^{-1}G_{\max}\}e^{-0.25\nu(t-t_k)} + 2\nu^{-1}G_{\max}\},\end{aligned}$$

for all $t \in [t_k, t_{k+1})$, $k = 0, 1, \dots$. It is worth to stress that these estimates hold independently on the number of impulses (finite or infinite). The differential equation (8) repeats the form of the system from Lemma 1, additionally from the theorem conditions and the facts established above all conditions of Lemma 1 are satisfied, that implies boundedness of the variable $\hat{\theta}$ with the following upper estimate for all $t \geq 0$:

$$|\tilde{\theta}(t)| \leq |C|||\Omega||\sqrt{\alpha}[e^{-0.5\gamma\alpha^{-1}t}|\tilde{\theta}(0)| + \alpha|C|\,||\Omega||(||C\delta(t) + v(t)||)].$$

Finally, it is required to prove the error e boundedness. This property follows by analysis of the equations (6), (9) in a manner similar to that performed for the variables Ω and δ (this approach allows the upper estimates on the hybrid behavior of the state estimation error to be computed). More simple way consists in observation that $e = \delta - \Omega\tilde{\theta}$, where all variables in the right hand side are bounded, anyway the inequalities hold for all $k = 0, 1, \dots$

$$\begin{aligned}|e(t_{k+1}^{+})| \leq & \omega(P)\{\rho^{0.5(k+1)}|e(t_0)| + \sqrt{2/(1-\rho)}[||r|| \\ & + \sqrt{2\mu}\nu^{-1}Z(||\tilde{\theta}||, ||\delta||, ||\Omega||, ||v||, ||p||)]\},\end{aligned}$$

and for all $t \in [t_k, t_{k+1})$, $k = 0, 1, \dots$

$$\begin{aligned}|e(t)| \leq & \omega(P)^2[\rho^{0.5k}|e(t_0)| + \sqrt{2/(1-\rho)}||r||]e^{-0.25\nu(t-t_k)} \\ & + 2\omega(P)\nu^{-1}Z(||\tilde{\theta}||, ||\delta||, ||\Omega||, ||v||, ||p||)(1 + \omega(P)\sqrt{\mu/(1-\rho)}e^{-0.25\nu(t-t_k)}).\end{aligned}$$

Part (i) of the theorem has been proven. From these estimates, asymptotically the following series of relations hold (independently on finite or infinite number of impulses):

$$\begin{aligned}\lim_{t\to+\infty}|\tilde{\theta}(t)| \leq & \alpha^{1.5}|C|^2 \lim_{T\to+\infty}||\Omega||^2_{[T,+\infty)}\{|C|||\delta||_{[T,+\infty)} + ||v||\} \\ \leq & \alpha^{1.5}|C|^2 D(||r||, ||p||, ||v||); \\ \lim_{t\to+\infty}|e(t)| \leq & Q(||r||, ||p||, ||v||).\end{aligned}$$

Substitution in these limit inequalities the upper estimates for p and r gives the inequalities from the part (ii) of the theorem. □

The proven theorem proposes the conditions under which the posed problem of an impulsive adaptive observer design is solved. In addition, the asymptotic estimates are calculated for state and parametric estimation errors (some other useful estimates are derived in the proof). The conditions of Theorem 1 imply that the continuous dynamics (the system (4)) is asymptotically stable ($\nu > 0$), while the discrete dynamics (5) could be stable with $\mu \leq 1$ or unstable with $\mu > 1$ under the stability restriction $\rho = \mu e^{-0.5\nu\tau_D} < 1$ (this condition regulates the frequency of generation of pulses in (4), (5)). The stability/instability of (4) and (5) are determined by the choice of L and K respectively. If $\mu > 1$, then the overall hybrid system stability is ensured by a sufficiently long obligatory activation of the continuous dynamics (4) increasing the dwell-time value τ_D. The case $\nu \leq 0$ can be considered in the same way under the condition $\mu < 1$, a simple modification of Theorem 1 proof is needed only.

In the case $\mu > 1$ the discrete dynamics (5) excites its continuous-time counterpart (4). As it will be shown below, such strategy could be rather useful improving persistency of excitation in the systems. From (5) the choice $\mu > 1$ can always provide the fulfillment of PE condition for the signal $\Omega(t)^T C^T$. Contrarily the conventional case [10, 32, 33], here the PE property $\Omega(t)^T C^T$ may not follows by the same property of the signal $G[y_v(t)]^T$. This is why in Theorem 1 the PE property is required for both of them.

Corollary 1. *Let all conditions of Theorem 1 hold and $v(t) = d(t) \equiv 0$ for all $t \geq 0$, then*

$$\lim_{t\to+\infty} |\theta - \hat{\theta}(t)| \leq aF_{\max}||x||, \lim_{t\to+\infty} |x(t) - z(t)| \leq bF_{\max}||x||,$$

where

$$\begin{aligned} a =&\chi^2 G_{\max}^2 |C| \omega(P)^2 \sqrt{2/(1-\rho)}, \\ b =&\{1 + \chi^3 (G_{\max}|C|)^2 \gamma [1 + \alpha^{1.5}\gamma^{-1}|C|G_{\max} \\ &+ \alpha^{1.5}(\chi G_{\max})^3 |C|^3]\}\omega(P)^2 \sqrt{2/(1-\rho)}. \end{aligned}$$

Proof. The claim follows by explicit substitution of $||v|| = ||d|| = 0$ in the estimates from Theorem 1. □

Therefore, the impulsive adaptive observer (4), (5) has a steady-state error proportional to the amplitude $F_{\max}||x||$. To solve this accuracy problem one can use the matrix function $F(y_v) = I_n$ as it has been done in [29], in this case $F_{\max} = 0$ and the exact asymptotic state and parameters estimation is guaranteed in (4), (5):

$$\lim_{t\to+\infty} |\theta - \hat{\theta}(t)| = 0, \lim_{t\to+\infty} |x(t) - z(t)| = 0.$$

However, in the case $F(y_v) = I_n$ it is rather difficult to ensure the Schur stability of the matrix $F(y_v) - K(y_v)C$ (actually it is possible if the whole state vector of (1) is available for measurements, i.e. $C = I_n$). However instability of $F(y_v) - K(y_v)C$ is not a restriction if the system excitation is needed, see Section 4 below, where for $F(y_v) = I_n$ and an unstable matrix $F(y_v) - K(y_v)C$ the possibility to improve PE property is discussed.

3.4 *Simplified case*

The matrix function inequalities, formulated in Theorem 1 to calculate the observer gains L, K, F, are rather hard to solve (since they depend on the variable y_v). Introducing supplementary constraints on the form of the system (1) it is possible to formulate these conditions in terms of LMIs.

Theorem 2. *Let Assumption 1 be satisfied and $A(y) = A$ in (1) and (4). Let the sequence of instants t_k, $k = 0, 1, \ldots$ have the dwell-time $\tau_D > 0$ and the signals $G[y_v(t)]^T$ and $\Omega(t)^T C^T$ be (ℓ, ϑ)–PE for some $\ell > 0$, $\vartheta > 0$. If there exist a matrix $P = P^T > 0$ with dimension $n \times n$ and $L(y_v) = L$, $K(y_v) = K$, $F(y_v) = F$ such that:*

$$[A - LC]^T P + P[A - LC] \leq -\nu P, \nu > 0,$$
$$[F - KC]^T P[F - KC] \leq 0.5\mu P, \ \mu > 0,$$

with $0 < \rho < 1$ where $\rho = \mu e^{-0.5\nu\tau_D}$, then:

1. *for any $x_0 \in X$, $z_0 \in \mathbb{R}^n$, $\Omega_0 \in \mathbb{R}^{n\times q}$, $\hat{\theta}_0 \in \mathbb{R}^q$ the solutions of the system (1)-(5) are bounded for all $t \geq 0$;*

2. *the following asymptotic estimates are satisfied:*

$$\lim_{t\to+\infty} |\theta - \hat{\theta}(t)| \leq \alpha^{1.5}|C|^2 S(||x||, ||v||, ||d||, |\theta|),$$
$$\lim_{t\to+\infty} |x(t) - z(t)| \leq H(||x||, ||v||, ||d||, |\theta|),$$

where

$$S(x, v, d, q) = D(|I_n - F|x + |K|v, \{L_\phi + L_G q + |L|\}v + d, v),$$
$$H(x, v, d, q) = Q(|I_n - F|x + |K|v, \{L_\phi + L_G q + |L|\}v + d, v),$$

and the functions $Q(r, p, v)$, $D(r, p, v)$ have been defined in Theorem 1.

Proof. Dynamics of the state estimation error $e = x - z$, the parametric estimation error $\tilde{\theta} = \theta - \hat{\theta}$ and the auxiliary error $\delta = e + \Omega\tilde{\theta}$ can be presented for all $t \in [t_{k-1}, t_k)$, $k = 1, 2, \ldots$ in the following form:

$$\dot{e}(t) = \{A - LC\}e(t) + G[y_v(t)]\tilde{\theta}(t) - \Omega(t)\dot{\tilde{\theta}}(t) + p(t), \quad (13)$$
$$p(t) = \phi[y(t), u(t)] - \phi[y_v(t), u(t)] + \{G[y(t)] - G[y_v(t)]\}\theta - Lv(t) + d(t);$$
$$\dot{\delta}(t) = \{A - LC\}\delta(t) + p(t); \quad (14)$$
$$\dot{\tilde{\theta}}(t) = \gamma\Omega(t)^T C^T\{Ce(t) + v(t)\} \quad (15)$$
$$= \gamma\Omega(t)^T C^T\{C\delta(t) - C\Omega(t)\tilde{\theta}(t) + v(t)\},$$

where $p \in \mathbb{R}^n$ is a new disturbing input as before. Under Assumption 1 the property

$$||p|| \leq \{L_\phi + L_G|\theta| + |L|\}||v|| + ||d||,$$

holds, and the equation (15) is actually valid for all $t \geq 0$ since the variable $\hat{\theta}$ has continuous dynamics only. The errors e and δ have impulse driven discrete dynamics, that from (5) can be written as follows for any t_k, $k = 0, 1, \ldots$ (note that $x(t_k) = x(t_k^+)$):

$$\begin{aligned} e(t_k^+) = x(t_k^+) - z(t_k^+) =& x(t_k) - Fz(t_k) - K\{y_v(t_k) - Cz(t_k)\} && (16)\\ =&\{F - KC\}e(t_k) + r(t_k),\\ r(t_k) =&(I_n - F)x(t_k) - Kv(t_k),\ ||r|| \le |I_n - F|||x|| + |K|||v||;\\ \delta(t_k^+) =& e(t_k^+) + \Omega(t_k^+)\tilde{\theta}(t_k^+) = \{F - KC\}\delta(t_k) + r(t_k), && (17) \end{aligned}$$

where $r \in \mathbb{R}^n$ is another disturbing bounded input. To prove boundedness of Ω and δ consider the Lyapunov function $V(\delta) = \delta^T P \delta$ for these variables (for brevity the variable δ is analyzed below). From (14)

$$\begin{aligned} \dot{V}(t) =&\delta(t)^T \left(\{A - LC\}^T P + P\{A - LC\}\right) \delta(t) + 2\delta(t)^T Pp(t)\\ \le& -\nu V(t) + 2\delta(t)^T Pp(t) \le -0.5\nu V(t) + 2\nu^{-1} p(t)^T Pp(t) \end{aligned}$$

for all $t \in [t_{k-1}, t_k)$, $k = 1, 2, ...$, which gives the estimate (11) in the time domain as in Theorem 1 proof. According to (17) for all t_k, $k = 0, 1, ...$ the impulse based behavior of the function $V(\delta)$ yields:

$$\begin{aligned} V(t_k^+) = V[\delta(t_k^+)] =& 2\delta(t_k)^T \{F - KC\}^T P\{F - KC\}\delta(t_k) + 2r(t_k)^T Pr(t_k)\\ \le& \mu V(t_k) + 2r(t_k)^T Pr(t_k) \le \mu V(t_k) + 2\lambda_{\max}(P)|r(t_k)|^2. \end{aligned}$$

Therefore, on the closed interval $[t_k, t_{k+1}]$, $k = 0, 1, ...$ the Lyapunov function V transformation is described by the expression (12). The rest part of the proof coincides with the same from Theorem 1.

The main difference between theorems 1 and 2 conditions consists in the use of matrix inequalities in Theorem 2, these inequalities can be represented as LMIs, then they can be resolved with respect to P, L, K, F. The conditions of existence of a transformation of (1) to a form with $A(y) = A$ are given in [25].

4 PERSISTENCY OF EXCITATION

Another difference between the theorems 1 and 2 could be in the use of the PE condition. Indeed, according to (4) the variable Ω dynamics is described by a linear time-invariant system in the case of Theorem 2, then the PE properties of the signals $\Omega(t)^T$ and $\Omega(t)^T C^T$ follow by the input excitation [11, 12]. The proof of this claim for the matrix variables and impulsive dynamics is given in the next subsection using several lemmas. The PE condition improvement in impulsive adaptive observers is discussed in the second subsection.

In this section we will always assume that $A(y) = A$ and $G(t) = G[y_v(t)]$ is bounded.

4.1 *Input-output PE in the impulsive linear systems*

The first two lemmas deal with the case of continuous dynamics (4) only.

Lemma 2. *Let the matrix $A - LC$ be Hurwitz. The signal $G(t)^T$ is (ℓ, ϑ)–PE for some $\ell > 0$, $\vartheta > 0$ if and only if there exist some $\ell' > 0$, $\vartheta' > 0$ such that the signal $\Omega(t)^T$ is (ℓ', ϑ')–PE in (4).*

Proof. Since the matrix $A-LC$ is Hurwitz, the system (4) solutions are globally defined and bounded for all $t \in \mathbb{R}$. By contrary, if the signal $G(t)^T$ is not PE, then for some vector $\zeta \in \mathbb{R}^q$, all $t \in \mathbb{R}$ and any $\varepsilon \geq 0$ there exists $\mathcal{L} > 0$ such that

$$\int_t^{t+\mathcal{L}} \zeta^T G(\tau)^T G(\tau)\zeta d\tau \leq \varepsilon.$$

This property implies that the vector signal $G(t)\zeta$ is asymptotically convergent. Multiplying the equation (4) by the vector ζ we obtain:

$$\dot{\psi} = \{A - LC\}\psi - G(t)\zeta, \ \psi = \Omega\zeta, \tag{18}$$

therefore, the signal $\psi(t)$ is also asymptotically convergent. Then for all $t \in \mathbb{R}$ and any $\varepsilon \geq 0$ there is $\mathcal{L}' > 0$ such that

$$\int_t^{t+\mathcal{L}'} \zeta^T \Omega(\tau)^T \Omega(\tau)\zeta d\tau = \int_t^{t+\mathcal{L}'} \psi(\tau)^T \psi(\tau) d\tau \leq \varepsilon.$$

Consequently, the signal $\Omega(t)^T$ also is not PE.

Now, let the signal $\Omega(t)^T$ be not PE, then for some vector $\zeta \in \mathbb{R}^q$, for all $t \in \mathbb{R}$ and any $\varepsilon \geq 0$ there is $\mathcal{L} > 0$ such that

$$\int_t^{t+\mathcal{L}} \zeta^T \Omega(\tau)^T \Omega(\tau)\zeta d\tau \leq \varepsilon.$$

Owing to this inequality, the input $G(t)\zeta$ asymptotic convergence follows from (18) and stability of the matrix $A - LC$:

$$\psi(t) = e^{(A-LC)t}\psi(0) + \int_0^t e^{(A-LC)(t-\tau)} G(\tau)\zeta d\tau.$$

Then, for all $t \in \mathbb{R}$ and any $\varepsilon \geq 0$ there is $\mathcal{L}' > 0$ such that

$$\int_t^{t+\mathcal{L}'} \zeta^T G(\tau)^T G(\tau)\zeta d\tau \leq \varepsilon$$

and the signal $G(t)^T$ is not PE. We have proven that if one of the signals $\Omega(t)^T$ or $G(t)^T$ is not PE, then anther one fails to possess this property, and *vice versa*. Negation of this result gives the lemma claim. □

Lemma 3. *Let the matrix $A-LC$ be Hurwitz, the pair of matrices $(A-LC, C)$ be observable and the signal $G(t)^T$ be (ℓ, ϑ)–PE for some $\ell > 0$, $\vartheta > 0$. Then there are some $\ell' > 0$, $\vartheta' > 0$ such that $\Omega(t)^T C^T$ is (ℓ', ϑ')–PE in (4).*

Proof. Let

$$R = \begin{bmatrix} C \\ C(A-LC) \\ \vdots \\ C(A-LC)^{n-1} \end{bmatrix},$$

be the observability matrix that has rank n by the lemma conditions. Define the new variable $\Theta = R\Omega$ and $S = R(A-LC)R^{-1}$, then the system (4) dynamics can be presented in the observer canonical form:

$$\dot{\Theta} = S\Theta + RG(t),\ t \in \mathbb{R}. \tag{19}$$

Since the matrix R has full rank, applying the same arguments as for the proof of Lemma 2 we can substantiate that, the signal $G(t)^T$ is (ℓ, ϑ)-PE if and only the signal $\Theta(t)^T$ is (ℓ', ϑ')-PE for some $\ell > 0$, $\vartheta > 0$, $\ell' > 0$, $\vartheta' > 0$. Since the signal $C\Omega(t)$ is represented by the first elements of the signal $\Theta(t)$, the claim follows. □

The next lemma analyzes PE conditions in the impulse driven system (4), (5).

Lemma 4. *Let the sequence of instants t_k, $k = 0, 1, ...$ have the dwell-time $\tau_D > 0$ and there exist matrices $P = P^T > 0$, L, K, F such that:*

$$[A-LC]^T P + P[A-LC] \le -\nu P, \nu > 0,$$
$$[F-KC]^T P[F-KC] \le \mu P,$$

with $0 < \rho < 1$ where $\rho = \mu e^{-0.5\nu\tau_D}$. The signal $G(t)^T = G[y_v(t)]^T$ is (ℓ, ϑ)–PE for some $\ell > 0$, $\vartheta > 0$ if and only if there exist some $\ell' > 0$, $\vartheta' > 0$ such that the signal $\Omega(t)^T$ is (ℓ', ϑ')–PE in (4), (5).

Proof. Again assume that the signal $G(t)^T$ is not PE, then for some vector $\zeta \in \mathbb{R}^q$, all $t \in \mathbb{R}$ and any $\varepsilon \ge 0$ there exists $\mathcal{L} > 0$ such that

$$\int_t^{t+\mathcal{L}} \zeta^T G(\tau)^T G(\tau)\zeta d\tau \le \varepsilon.$$

This property implies that the vector signal $G(t)\zeta$ is asymptotically convergent. Multiplying the equation (4), (5) by the vector ζ we obtain for $\psi = \Omega\zeta$ and $f(t) = G(t)\zeta$:

$$\dot{\psi}(t) = \{A-LC\}\psi(t) - f(t),\ t \in [t_k, t_{k+1}),\ k = 0, 1, ...; \tag{20}$$
$$\psi(t_k^+) = \{F-KC\}\psi(t_k), k = 1, 2, ... \tag{21}$$

Consider the Lyapunov function $V(\psi) = \psi^T P\psi$, for all $t \in [t_{k-1}, t_k)$, $k = 1, 2, ...$ from (20) we have:

$$\begin{aligned}\dot{V}(t) =& \psi(t)^T \left(\{A-LC\}^T P + P\{A-LC\}\right)\psi(t) - 2\psi(t)^T Pf(t)\\ \le& -\nu V(t) - 2\psi(t)^T Pf(t) \le -0.5\nu V(t) + 2\nu^{-1} f(t)^T Pf(t),\end{aligned}$$

that gives the following estimate in the time domain:

$$V(t) \leq V(t_k)e^{-0.5\nu(t-t_k)} + 4\nu^{-2}\lambda_{\max}(P)||f||^2_{[t_k,t_{k+1})}. \tag{22}$$

According to (21) for all t_k, $k = 0, 1, ...$ the impulse based behavior of the function $V(\psi)$ yields:

$$V(t_k^+) = V[\psi(t_k^+)] = \psi(t_k)^T\{F - KC\}^T P\{F - KC\}\psi(t_k) \leq \mu V(t_k).$$

Therefore, on the closed interval $[t_k, t_{k+1}]$, $k = 0, 1, ...$ the Lyapunov function V transformation is described by the following expression:

$$V(t_{k+1}^+) \leq \mu[V(t_k)e^{-0.5\nu(t_{k+1}-t_k)} + 4\nu^{-2}\lambda_{\max}(P)||f||^2_{[t_k,t_{k+1})}]. \tag{23}$$

Applying the estimate (23) recurrently $k+1$ times for $k = 0, 1, ...$ we obtain ($0 < \rho = \mu e^{-0.5\nu\tau_D} < 1$):

$$\begin{aligned} V(t_{k+1}^+) \leq& \rho^{k+1}V(t_0) + 4\lambda_{\max}(P)\nu^{-2}\mu||f||^2_{[t_0,t_{k+1}]}\sum_{i=0}^{k}\rho^i \\ \leq& \rho^{k+1}V(t_0) + 4\lambda_{\max}(P)(1-\rho)^{-1}\nu^{-2}\mu||f||^2_{[t_0,t_{k+1}]}. \end{aligned}$$

Taking in mind (22) for all $k = 0, 1, ...$ and $t \in [t_k, t_{k+1})$ the estimate

$$\begin{aligned} V(t) \leq& \{\rho^{k+1}V(t_0) + 4\lambda_{\max}(P)(1-\rho)^{-1}\nu^{-2}\mu||f||^2_{[t_0,t_{k+1}]}\}e^{-0.5\nu(t-t_k)} \\ &+ 4\nu^{-2}\lambda_{\max}(P)||f||^2_{[t_k,t_{k+1})} \end{aligned}$$

holds, therefore for $0 < \rho < 1$ and $\nu < 0$ with a convergent input f the variable ψ is bounded and asymptotically converging (independently on the number of impulses finite or infinite). Then for all $t \in \mathbb{R}$ and any $\varepsilon \geq 0$ there exists $\mathcal{L}' > 0$ such that

$$\int_t^{t+\mathcal{L}'} \zeta^T\Omega(\tau)^T\Omega(\tau)\zeta d\tau = \int_t^{t+\mathcal{L}'} \psi(\tau)^T\psi(\tau)d\tau \leq \varepsilon.$$

Consequently, the signal $\Omega(t)^T$ also is not PE.

Conversely, let the signal $\Omega(t)^T$ be not PE, then for some vector $\zeta \in \mathbb{R}^q$, for all $t \in \mathbb{R}$ and any $\varepsilon \geq 0$ there exists $\mathcal{L} \geq 0$ such that

$$\int_t^{t+\mathcal{L}} \zeta^T\Omega(\tau)^T\Omega(\tau)\zeta d\tau \leq \varepsilon.$$

For the system (20), (21) this is equivalent to the input f asymptotic convergence. Indeed, the system (20) is time-invariant linear asymptotically stable system with the additive input f (as in Lemma 2 proof the convergences of ψ and f are strictly interconnected), inclusion of the discrete dynamics (21) gives for all $t \in [t_k, t_{k+1})$, $k = 0, 1, ...$:

$$\psi(t) = e^{(A-LC)(t-t_k)}\psi_k + \int_{t_k}^{t} e^{(A-LC)(t-\tau)}G(\tau)\zeta d\tau; \tag{24}$$

$$\psi_0 = \psi(0),\ \psi_k = (F - KC)\psi(t_k)\ \forall k = 1, 2, ...$$

Then the variable $\psi(t)$ convergence implies the same property for the convolution integral in (24). Therefore, for all $t \in \mathbb{R}$ and any $\varepsilon \geq 0$ there exists $\mathcal{L}' \geq 0$ such that

$$\int_t^{t+\mathcal{L}'} \zeta^T G(\tau)^T G(\tau)\zeta d\tau \leq \varepsilon$$

and the signal $G(t)^T$ is not PE. We have proven that if one of the signals $\Omega(t)^T$ or $G(t)^T$ is not PE, then anther one fails to possess this property in the hybrid system (4), (5). The claim of Lemma 4 follows by this result negation. □

If the pair of matrices $(A - LC, C)$ is observable, then combining lemmas 3 and 4 the PE property of the signal $\Omega(t)^T C^T$ can be deduced from PE of $G[y_v(t)]^T$ for the system (4), (5). Thus in the conditions of Theorem 2 it is enough to check the PE property for the signal $G[y_v(t)]^T$ only (provided that the pair of matrices $(A - LC, C)$ is observable).

4.2 *PE improvement in impulsive adaptive observers*

The last thing to show is that under some restrictions the hybrid system (4), (5) may ensure better excitation than the conventional continuous dynamics (4) alone. In order to show this, note that the PE condition for the signal $\Omega(t)^T C^T$ corresponds to strict positive definiteness of the following integral for some $\ell > 0$, $\vartheta > 0$ and all $t \in \mathbb{R}$:

$$\int_t^{t+\ell} \Omega(\tau)^T C^T C\Omega(\tau)d\tau \geq \vartheta I_q,$$

which left hand side can be interpreted as the L_2 norm of the signal $C\Omega(t)$ on interval $[t, t + \ell]$. The system (4) solution $\Omega(t)$ is described by

$$\Omega(t) = e^{(A-LC)(t-t_k)}\Omega(t_k) + \int_{t_k}^{t} e^{(A-LC)(t-\tau)}G[y_v(\tau)]d\tau$$

for all $t \in [t_k, t_{k+1})$. This solution has the decaying part proportional to the initial conditions $\Omega(t_k)$ and the forced part governed by $G[y_v(t)]$, that ensures the PE property for $\Omega(t)^T$. If the matrix $F - KC$ has all singular values bigger than 1 and $t_{k+1} - t_k \leq \ell$ for all $k = 0, 1, ...$, then

$$|\Omega(t_k^+)| = |(F - KC)\Omega(t_k)| > |\Omega(t_k)|$$

and the L_2 norm of the signal $C\Omega(t)$ has to be augmented accordingly for $t \geq t_k$. The dwell-time stability conditions developed in theorems 1 and 2 ensure the overall system stability in this case.

This proposed strategy has the following interpretation: if a designer cannot excite explicitly the system (1) (improving the PE property), then it is possible to excite the observer equations by proper impulses, which also leads to amplification of the level of excitation and quality of the parameter estimation. Thus the hybrid system (4), (5) improves PE abilities of the signal $G[y_v(t)]$. Illustrations of this conclusion are given below on four examples.

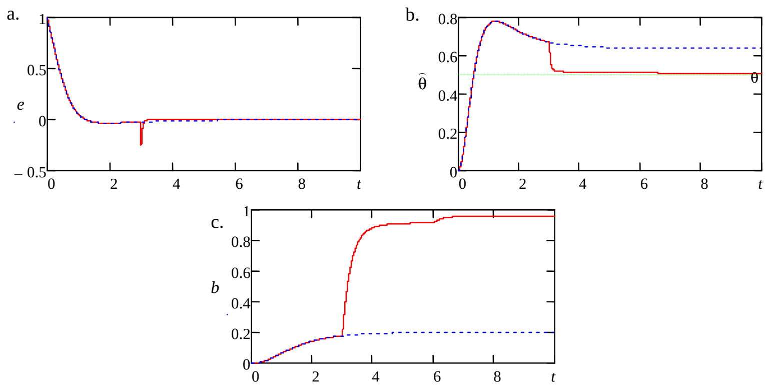

Fig. 1: The results of simulation for the linear example

5 SIMULATION

In this section we will analyze three examples. The first one (a simple linear first order system) is given to clearly demonstrate advantages and peculiarities of the proposed observer. The next two examples are nonlinear systems of the third and fourth orders respectively.

5.1 A linear system

Consider a conventional linear benchmark example in the adaptive control theory:

$$\dot{x} = -x + \theta x, y = x.$$

We will assume that the system is stable and therefore $\theta < 1$. Then $x(t)$ is exponentially converging and the system has no PE property. The results of application of the conventional adaptive observer [10, 32, 33] with $L(y) = 1$, $\gamma = 10$ to this system are shown in Fig. 1 by the blue dash lines. In Fig. 1,a the estimation error is plotted, in Fig. 1,b the variable $\hat{\theta}(t)$ is presented, an estimation of the PE property is shown in Fig. 1,c:

$$b(t) = \int_0^t \Omega(\tau)^T \Omega(\tau) d\tau.$$

The same trajectories for the impulsive adaptive observer (4), (5) with $L(y) = 1$, $K(y) = -10$, $F(y) = 1$ and $\tau_D = 3$ are shown in Fig. 1 by the red solid lines. As it is possible to conclude from these results, the application of additional impulsive feedbacks with unstable $F - KC$ leads to PE improvement.

5.2 Lorenz chaotic model

Another example deals with the Lorenz system observation [10]:

$$\dot{x}_1 = \sigma(x_2 - x_1); \dot{x}_2 = (r + \theta)x_1 - x_2 - x_1 x_3; \dot{x}_3 = -\beta x_3 + x_1 x_2, y = x_1,$$

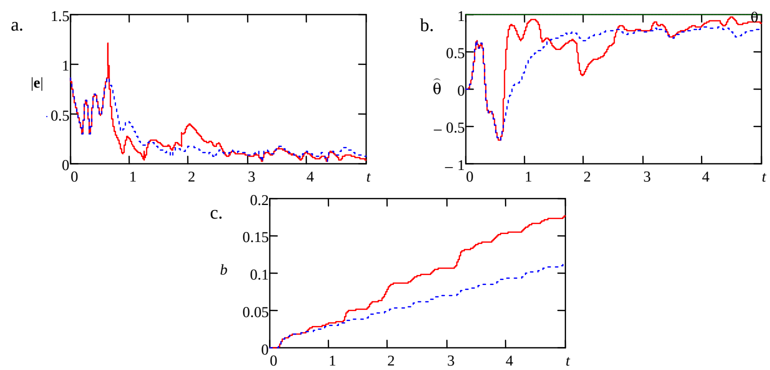

Fig. 2: The results of simulation for the Lorenz system

which for $\sigma = 10$, $\beta = 8/3$, $r = 97$, $\theta = 0$ demonstrates a chaotic behavior. For some small $|\theta|$ the system still has a similar behavior, that implies PE property for the Lorenz system on the strange attractor. The results of application of the conventional adaptive observer [10, 32, 33] with $L(y) = [0\ r + \sigma\ 0]^T$, $\gamma = 100$ to this system are shown in Fig. 2 by the blue dash lines. In Fig. 2,a the estimation error norm $|e(t)|$ is plotted, in Fig. 2,b the variable $\hat{\theta}(t)$ is presented, the estimation $b(t)$ of the PE property is shown in Fig. 2,c. The same trajectories for the impulsive adaptive observer (4), (5) with $L(y) = [0\ r+\sigma\ 0]^T$, $K(y) = -1.5[1\ 1\ 1]^T$, $F(y) = I_3$ and $\tau_D = 0.625$ are shown in Fig. 2 by the red solid lines (in this case $P = I_3$ in Theorem 1, all conditions of this theorem are satisfied). As it is possible to conclude from these results, the application of additional impulsive feedbacks with unstable matrix $F - KC$ leads to PE property improvement. However, in this case since the PE property is strongly presented in the system such improvement is not significant.

5.3 A single-link flexible joint robot

The third example deals with a single-link flexible joint robot estimation [5, 30], where due to joint flexibility the system nonlinearities are modeled as a stiffening torsional spring and the gravitational force. Denoting by φ_m, ω_m, φ_l and ω_l, the motor and link position and velocities respectively, the equations are given by

$$\dot{\varphi}_m = \omega_m,\ \dot{\omega}_m = J_m^{-1}(\tau(\varphi_m, \varphi_l) - B\omega_m + K_\tau u),\ y_1 = \varphi_l;$$
$$\dot{\varphi}_l = \omega_l,\ \dot{\omega}_l = -J_l^{-1}(\tau(\varphi_m, \varphi_l) + mgh\sin(\varphi_l)),\ y_2 = \varphi_m,$$

where J_m is the inertia of the motor, J_l is the inertia of the link, $2h$ and m represent the length and mass of the link, B is the viscous friction, and K_τ is the amplifier gain; $\tau(\varphi_m, \varphi_l) = \theta_1(\varphi_l - \varphi_m) + \theta_2(\varphi_l - \varphi_m)^3$ is the load torque, the coefficients θ_1, θ_2 are unknown. The physical values of parameters are the

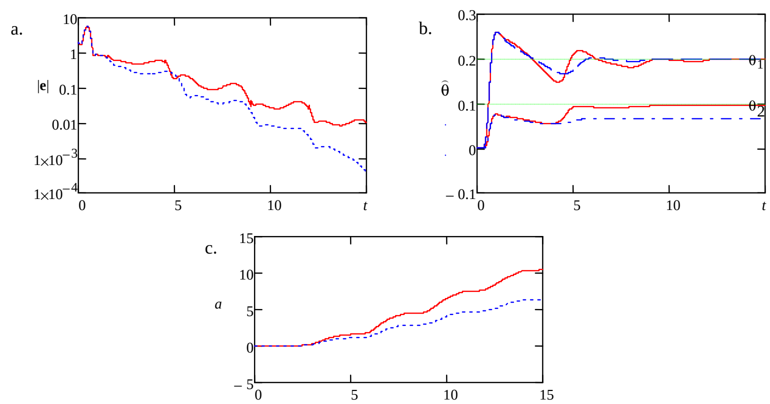

Fig. 3: The results of simulation for the robot example

following

$$J_m = 3.7 \times 10^{-3},\ J_l = 9.3 \times 10^{-3},\ h = 0.15,\ m = 0.21,\ B = 0.046,$$
$$K_\tau = 0.08,\ g = 9.8,\ \theta_1 = 0.2,\ \theta_2 = 0.1.$$

The control $u(t) = 0.1\sin(t)$ ensures a weak PE property in the system. The results of application of the conventional adaptive observer [10, 32, 33] with

$$L(y) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}^T, \gamma = 0.1$$

to this system are shown in Fig. 3 by the blue dash lines. In Fig. 3,a the estimation error norm $|e(t)|$ is plotted in logarithmic scale, in Fig. 3,b the variable $\hat{\theta}(t)$ is presented, the estimation $a(t) = \lambda_{\min}(b(t))$ of the PE property is shown in Fig. 3,c. The same trajectories for the impulsive adaptive observer (4), (5) with

$$L(y) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}^T, K(y) = -0.5\begin{bmatrix} 1 & 0.1 & 1 & 0.1 \\ 0.1 & 0 & 0.1 & 0 \end{bmatrix}^T,$$

$F(y) = I_4$ and $\tau_D = 1.5$ are shown in Fig. 3 by the red solid lines (in this case $P = I_4$ in Theorem 2, all conditions of this theorem are satisfied). As it is possible to conclude from these results the additional impulsive feedbacks with unstable matrix $F - KC$ leads to PE property improvement and a better estimation.

6 CONCLUSION

The paper presents an approach for impulsive adaptive observer design. The additional impulsive feedback is easy to implement complementary to the scheme of conventional adaptive observers proposed in [10, 32, 33, 3]. Different stability conditions are proposed, some of them assumes stability of the continuous-time loop of the observer and instability of the impulsive feedbacks. It is shown that such unstable impulses additionally excite the observer dynamics improving PE property of the system (in all cases the overall stability of the hybrid system is guaranteed). In other words, the idea of the approach is as follows: if it is not possible to excite explicitly the system (improving the PE property directly), then it is possible to excite the observer equations augmenting the level of excitation available for identification. The efficiency of the proposed approach is demonstrated by computer simulations for three examples. On examples of weakly excited systems the impulsive adaptive observer has shown superior results over the conventional one, for chaotic systems (which dispose a proper sufficient level of excitation) the improvement introduced by impulsive adaptive observer is minor but remarkable.

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